First Part: Basic Algebraic Logic

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Outline of this tutorial

1. The basics of algebraic logic
2. Residuated lattices
3. Substructural logics
4. The interplay of algebra and logic

Expanded version of these notes

G. Metcalfe, F. Paoli, C. Tsinakis, "Ordered algebras and logic", in H. Hosni and F. Montagna, *Probability, Uncertainty, Rationality*, Pisa, Edizioni della Normale, pp. 1-85, 2010.

George Boole (1)

Boole's *Mathematical Analysis of Logic* (1847) and *An Investigation of the Laws of Thought (1854)* mark the official birth of modern mathematical logic. Boole admittedly sought no less than to "investigate the fundamental laws of those operations of the mind by which reasoning is performed, to give expression to them in the symbolical language of a calculus, and upon this foundation to establish the science of Logic and construct its method."

Yet, Boole did not depart from tradition as radically as it might seem: about one third of his *Mathematical Analysis of Logic* is occupied by an algebraic treatment of Aristotelian syllogistic logic. For Boole, ordinary logic is concerned with assertions about *classes* of objects. He then translated the latter into equations in the language of classes. His approach contained numerous errors, partly due to his insistence that the algebra of logic should behave like ordinary algebra.

George Boole (2)

Although the name "Boolean algebra" might suggest that the inventor of this concept was Boole, there is by now widespread agreement that he was not. Not that it is always easy to clearly understand what Boole had in mind when working on his calculus of classes. As T. Hailperin puts it,

> *[Boole carried out] operations, procedures, and processes of an algebraic character, often inadequately justified by present-day standards and, at times, making no sense to a modern mathematician. [...] Boole considered this acceptable so long as the end result could be given a meaning.*

Boole's operations of *combination* xy of two classes x and y and of *aggregation* $x + y$ of x and y do correspond to intersection and union, respectively, but the latter only makes sense when x and y are disjoint. Boole's algebras bear therefore some resemblance to *partial algebras*, except that often Boole found it unobjectionable to disregard his disjointness condition throughout his calculations, provided the final result did not violate the condition itself.

Stanley Jevons

Boole preceded an array of researchers who tried to develop further his idea of turning logical reasoning into an algebraic calculus. Stanley Jevons (1835-1882), Charles S. Peirce (1839-1914), and Ernst Schröder (1841-1902) took their cue from Boole's investigations, but suggested improvements and modifications to his work.

Jevons was dissatisfied with Boole's choice of primitive set-theoretical operations. He did not like the fact that aggregation was a partial operation. In his *Pure Logic* (1864), he suggested a variant of Boole's calculus which he proudly advertised as based only on "processes of self-evident meaning and force." He viewed $+$ as a total operation making sense for any pair of classes, essentially corresponding to set-theoretic union. He also showed that all the expressions he used remained interpretable throughout the intermediate steps of his calculations, thereby overcoming one of the main drawbacks of Boole's work.

Charles Sanders Peirce

Peirce is usually credited with the foundation of the algebra of relations, for which Schröder also made significant developments. However, in the 12,000 pages of his published work – rising to an astounding 90,000 if we take into account his unpublished manuscripts – much more can be found:

- He investigated the laws of propositional logic, discovering that all the usual propositional connectives were definable in term of the single connective NAND.
- he introduced quantifiers, although, unlike Frege, he did not go so far as to suggest an axiomatic calculus for quantified logic.
- He conceived of complex and fascinating graphs by which he could represent logical syntax in two or even three dimensions.

For all these achievements, however, his impact on logic would not be even remotely comparable to that exerted by Frege.

Adolf Lindenbaum and Alfred Tarski

Roughly at the same time as Hilbert and Bernays pinned down their standard presentation of first order classical logic, E.L. Post (1897-1954), J. Łukasiewicz (1878-1956), and C.I. Lewis (1883-1964), among others, introduced the first Hilbert-style calculi for some propositional (many-valued or modal) non-classical logics. Thus, already in the 1930's classical logic had quite a number of competitors (including intuitionistic logic), each one trying to capture a different concept of *logical consequence*.

But what should count, abstractly speaking, as a concept of logical consequence? In answering this question, the Polish logicians Adolf Lindenbaum (1904-1941) and Alfred Tarski (1901-1983) initiated a confluence of the Fregean and algebra of logic traditions into one unique stream. Lindenbaum and Tarski showed how it is possible to associate in a canonical way propositional logical calculi (and their attendant consequence relations) with classes of algebras.

Propositional languages

A (propositional) *language* over a countably infinite set X, whose members are referred to as *variables*, is a nonempty set $\mathcal{L}$ (disjoint from X), whose members are called *connectives*, such that a nonnegative integer n is assigned to each member c of $\mathcal{L}$. This integer is called the *arity* of c. The set Fm of $\mathcal{L}$-*formulas* over X is defined as follows:

- *Inductive beginning*: Every member p of X is a formula.
- *Inductive step*: If c is a connective of arity n and $\alpha_1, ..., \alpha_n$ are formulas, then so is $c(\alpha_1, ..., \alpha_n)$.

We confine ourselves to cases where $\mathcal{L}$ is finite. For *binary* connectives the customary infix notation will be employed.

The formula algebra

If $\mathcal{L} = \{c_1, ..., c_n\}$ is a language over X, then by the inductive definition of formula

$$\mathbf{Fm} = \langle Fm, c_1, ..., c_n \rangle$$

is an algebra of type $\mathcal{L}$, called the *formula algebra* of $\mathcal{L}$.
Given a formula $\alpha(p_1, ..., p_n)$ containing at most the indicated variables, an algebra $\mathbf{A}$ of language $\mathcal{L}$ and $a_1, ..., a_n \in A$, $\alpha^{\mathbf{A}}(a_1, ..., a_n)$ (or $\alpha^{\mathbf{A}}(\overrightarrow{a})$) is the result of the application to α of the unique homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ such that $h(p_i) = a_i$ for all $i \leq n$.
An *equation* of language $\mathcal{L}$ is a pair (α, β) of $\mathcal{L}$-formulas, written $\alpha \approx \beta$.
Endomorphisms on $\mathbf{Fm}$ are called *substitutions*, and α is a *substitution instance* of β in case there is a substitution σ s.t. $\alpha = \sigma(\beta)$.

Consequence relations

A *consequence relation* over the formula algebra **Fm** is a relation $\vdash \subseteq \wp(Fm) \times Fm$ with the following properties:

1. $\alpha \vdash \alpha$ *(reflexivity)*;
2. If $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha$ *(monotonicity)*;
3. If $\Gamma \vdash \alpha$ and $\Delta \vdash \gamma$ for every $\gamma \in \Gamma$, then $\Delta \vdash \alpha$ *(cut)*.

A consequence relation $\vdash$ is:

- *substitution-invariant* in case, if $\Gamma \vdash \alpha$ and σ is a substitution on **Fm**, then $\sigma(\Gamma) \vdash \sigma(\alpha)$ $(\sigma(\Gamma) = \{\sigma(\gamma) : \gamma \in \Gamma\})$;
- *finitary* in case, if $\Gamma \vdash \alpha$, there exists a finite $\Delta \subseteq \Gamma$ s.t. $\Delta \vdash \alpha$.

Propositional logics

A (propositional) *logic* is a pair $\mathrm{L} = (\mathbf{Fm}, \vdash)$, where $\mathbf{Fm}$ is the formula algebra of some given language $\mathcal{L}$ and $\vdash$ is a substitution-invariant consequence relation over $\mathbf{Fm}$.
Logic = a logical language + a concept of consequence among formulas of that language according to which:

- every formula follows from itself;
- whatever follows from a set of premises also follows from any larger set of premises;
- whatever follows from consequences of a set of premises also follows from the set itself;
- whether a conclusion follows or not from a set of premises only depends on the *logical form* of the premises and the conclusion themselves.

A formula α is a *theorem* of the logic $\mathrm{L} = (\mathbf{Fm}, \vdash)$ if $\emptyset \vdash \alpha$.

Hilbert-style calculi

An *inference rule* over **Fm** is a pair $\mathrm{R} = (\Gamma, \alpha)$, where Γ is a finite (possibly empty) subset of *Fm* and $\alpha \in Fm$. If Γ is empty, the rule is an *axiom*; otherwise, it is a *proper rule*.
A *Hilbert-style calculus* (over **Fm**) is a set of inference rules over **Fm** that contains at least one axiom and at least one proper rule.
For axioms, outer brackets and the empty set symbol are usually omitted; also, proper rules $(\{\alpha_1, \ldots, \alpha_n\}, \alpha)$ are written in the fractional form

$$\frac{\alpha_1, \ldots, \alpha_n}{\alpha}$$

An example: HCL

A1. $\alpha \to (\beta \to \alpha)$	
A2. $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$	
A3. $\alpha \wedge \beta \to \alpha$	A4. $\alpha \wedge \beta \to \beta$
A5. $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to \beta \wedge \gamma))$	
A6. $\alpha \to \alpha \vee \beta$	A7. $\beta \to \alpha \vee \beta$
A8. $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$	
A9. $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$	A10. $\alpha \to (\neg\alpha \to \beta)$
A11. $\neg\neg\alpha \to \alpha$	A12. $(\alpha \to \alpha) \to 1$
A13. $1 \to (\alpha \to \alpha)$	A14. $0 \to \neg 1$
A15. $\neg 1 \to 0$	R1. $\dfrac{\alpha \qquad \alpha \to \beta}{\beta}$

Derivations in Hilbert-style calculi

If $\Delta \cup \{\beta\} \subseteq Fm$ and HL is a Hilbert-style calculus over **Fm**, a *derivation* of β from Δ in HL is a finite sequence $\beta_1, ..., \beta_n$ of formulas in Fm s.t. $\beta_n = \beta$ and for each β_i $(i \leq n)$:

1. either β_i is a member of Δ; or
2. β_i is a substitution instance of an axiom of HL; or else
3. there are a substitution σ and an inference rule $(\Gamma, \alpha) \in \mathrm{HL}$ such that $\beta_i = \sigma(\alpha)$ and, for every $\gamma \in \Gamma$, $\sigma(\gamma) \in \{\beta_1, ..., \beta_{i-1}\}$.

Logics from Hilbert-style calculi

From any Hilbert-style calculus HL over **Fm** we can extract a logic $(\mathbf{Fm}, \vdash_{\mathrm{HL}})$ by specifying that $\Gamma \vdash_{\mathrm{HL}} \alpha$ whenever there is a derivation of α from Γ in HL.

Such logics are called *deductive systems* and are finitary (this much is clear from the very definition of derivation: after we prune Γ of all that is not necessary to derive α, we are left with a finite set).

Classical propositional logic can be now identified with the deductive system $\mathrm{CL} = (\mathbf{Fm}, \vdash_{\mathrm{HCL}})$.

What did Lindenbaum and Tarski prove?

Developing an idea by Lindenbaum, Tarski showed in 1935 in what sense Boolean algebras can be considered the algebraic counterpart of CL. Actually, Tarski pointed out a rather weak kind of correspondence between Boolean algebras and classical logic: he showed that the former are an *algebraic semantics* for the latter, a notion that we now proceed to explain in full generality.

Translations

Let $\mathrm{L} = (\mathbf{Fm}, \vdash_{\mathrm{L}})$ be a logic in the language $\mathcal{L}$, and let $\tau = \{\gamma_i(p) \approx \delta_i(p)\}_{i \in I}$ be a set of equations in a single variable of $\mathcal{L}$. We may also think of τ as a function which maps formulas in *Fm* to sets of equations of the same type.
Thus, we let $\tau(\alpha)$ stand for the set

$$\{\gamma_i(p/\alpha) \approx \delta_i(p/\alpha)\}_{i \in I}.$$

Now, let $\mathcal{K}$ be a class of algebras also of the same language. We say that $\mathcal{K}$ is an *algebraic semantics* for L if, for some such τ, the following condition holds for all $\Gamma \cup \{\alpha\} \subseteq Fm$:

$$\Gamma \vdash_{\mathrm{L}} \alpha \quad \text{iff} \quad \begin{array}{l} \text{for every } \mathbf{A} \in \mathcal{K} \text{ and every } \overrightarrow{a} \in A^n, \\ \text{if } \tau(\gamma)^{\mathbf{A}}(\overrightarrow{a}) \text{ for all } \gamma \in \Gamma, \text{ then } \tau(\alpha)^{\mathbf{A}}(\overrightarrow{a}), \end{array}$$

a condition which can be rewritten as

$$\Gamma \vdash_{\mathrm{L}} \alpha \quad \text{iff} \quad \{\tau(\gamma) : \gamma \in \Gamma\} \vdash_{Eq(\mathcal{K})} \tau(\alpha).$$

The intuitive idea

Given an algebra $\mathbf{A} \in \mathcal{K}$, valuations $h : \mathbf{Fm} \rightarrow \mathbf{A}$, where $\mathbf{A} \in \mathcal{K}$, are interpreted as "assignments of meanings" to elements of Fm; thus, elements of A can be seen as "meanings of propositions" or "truth values". The translation map τ defines, for every $\mathbf{A} \in \mathcal{K}$, a "truth set" $T \subseteq A$ in the following sense: $\mathbf{A}$ satisfies $\tau(\alpha)$ just in case every valuation of α on $\mathbf{A}$ maps it to a member of T (intuitively: the meaning of α in $\mathbf{A}$ belongs to the set of "true values" and hence is true).
We want α to follow from the set of premises Γ just in case, whenever we assign a "true" value to all the premises in some algebra in $\mathcal{K}$, the conclusion is also assigned a "true" value.
In particular, if $\mathcal{L}$ contains a nullary connective 1, it is possible to choose τ to be the singleton $\{p \approx 1\}$ and, in particular, the element $1^{\mathbf{A}}$ as "true".

Classical logic and Boolean algebras (1)

A *Boolean algebra* is usually defined as an algebra

$$\mathbf{A} = \langle A, \wedge, \vee, \neg, 1, 0 \rangle$$

such that $\langle A, \wedge, \vee, 1, 0 \rangle$ is a bounded distributive lattice and, for every $a \in A$, $a \wedge \neg a = 0$ and $a \vee \neg a = 1$.

To apply the definition of algebraic semantics, Boolean algebras must be algebras of the same language as the formula algebra of CL, whence it is expedient to include in the language the derived operation symbol $\rightarrow$ (defined via $p \rightarrow q = \neg p \vee q$).

We also set

$$\tau = \{p \approx 1\}$$

Classical logic and Boolean algebras (2)

Theorem

The class $\mathcal{BA}$ of Boolean algebras is an algebraic semantics for CL*.*

Proof.

(Sketch). The left-to-right implication can be established by induction on the length of a derivation of α from Γ in HCL: we show that axioms A1 to A15 are always evaluated at $1^{\mathbf{A}}$ for every $\mathbf{A} \in \mathcal{BA}$ and that the proper inference rule R1 preserves this property.

The converse implication is trickier. We show the contrapositive: we suppose that $\Gamma \nvdash_{\mathrm{HCL}} \alpha$ and prove that there exist a Boolean algebra $\mathbf{A}$ and a sequence of elements $\overrightarrow{a}$ such that $\gamma(\overrightarrow{a}) = 1^{\mathbf{A}}$ for all $\gamma \in \Gamma$, yet $\alpha(\overrightarrow{a}) \neq 1^{\mathbf{A}}$. □

Classical logic and Boolean algebras (3)

Proof.

Let T be the smallest set of $\mathcal{L}$-formulas that includes Γ and is closed under $\vdash_{\mathrm{HCL}}$. Define $\Theta_T \subseteq Fm^2$ as follows:

$$(\beta, \gamma) \in \Theta_T \text{ iff } \beta \to \gamma, \gamma \to \beta \in T.$$

We have to show that:

1. Θ_T is a congruence on $\mathbf{Fm}$, and the coset $\left[1^{\mathbf{Fm}}\right]_{\Theta_T}$ is just T;
2. the quotient $\mathbf{Fm}/\Theta_T$ is a Boolean algebra.

Proofs of (1) and (2) make heavy use of syntactic lemmas established for HCL.To get a falsifying model, it now suffices to take $\mathbf{A} = \mathbf{Fm}/\Theta_T$ (we are justified in so doing by (2)) and evaluate each p in $\Gamma \cup \{\alpha\}$ as its own congruence class modulo Θ_T: then $\gamma^{\mathbf{A}}(\overrightarrow{[p]_{\Theta_T}}) = 1^{\mathbf{A}}$ for all $\gamma \in \Gamma$ (since $\Gamma \subseteq T$) yet $\alpha^{\mathbf{A}}(\overrightarrow{[p]_{\Theta_T}}) \neq 1^{\mathbf{A}}$ (since $\alpha \notin T$). □

Algebraic logic after Lindenbaum and Tarski

Algebraic logic rapidly developed after World War Two, once again to the credit of Polish logicians.

Although Tarski had permanently settled in the States before that time, establishing in Berkeley what would become the leading research group in algebraic logic worldwide, his compatriots Jerzy Łoś, Roman Suszko, Helena Rasiowa, and Roman Sikorski kept the flag of Polish algebraic logic flying, developing in detail throughout the 1950's and 1960's the theory of *logical matrices* initiated twenty years earlier by Łukasiewicz and Tarski himself.

A major breakthrough came about in 1989, when Wim Blok and Don Pigozzi (one of Tarski's students) published their monograph on *algebraizable logics*, considered a milestone in the area of abstract algebraic logic.

Shortcomings of the Lindenbaum-Tarski approach (1)

First shortcoming: the relationship between a logic and its algebraic semantics can be promiscuous.

- *There are logics with no algebraic semantics*. An example is the deductive system $(\mathbf{Fm}, \vdash_{\mathrm{HI}})$, where HI is the Hilbert-style calculus whose sole axiom is $\alpha \to \alpha$ and whose sole inference rule is modus ponens.
- *The same logic can have more than one algebraic semantics*. By Glivenko's Theorem, CL admits not only $\mathcal{BA}$ as an algebraic semantics, but also the variety $\mathcal{HA}$ of *Heyting algebras*, by choosing $\tau = \{\neg\neg p \approx 1\}$.
- *There can be different logics with the same algebraic semantics*. Since Heyting algebras are an algebraic semantics for *intuitionistic logic* IL, the previous example shows that both CL and IL have $\mathcal{HA}$ as an algebraic semantics.

Shortcomings of the Lindenbaum-Tarski approach (2)

Second shortcoming: the relationship between a logic and its algebraic semantics can be asymmetric.
The property of belonging to the set of "true values" of an algebra $\mathbf{A} \in \mathcal{K}$ must be definable by means of the set of equations τ, whence the class $\mathcal{K}$ has the expressive resources to indicate when a given *formula* is valid in L.
On the other hand, the logic L need not have the expressive resources to indicate when a given *equation* holds in $\mathcal{K}$.

Shortcomings of the Lindenbaum-Tarski approach (3)

For example, there is no way in CL to express, by means of a condition involving a set of formulas, when it is the case that $\alpha \approx \beta$ holds in its algebraic semantics $\mathcal{HA}$. This makes a sharp contrast with the other algebraic semantics $\mathcal{BA}$: as we have seen, $\alpha \approx \beta$ holds in $\mathcal{BA}$ just in case $\vdash_{\mathrm{HCL}} \alpha \to \beta$ and $\vdash_{\mathrm{HCL}} \beta \to \alpha$.

The notion of *algebraizability* aims at making precise this stronger relation between a logic and a class of algebras which holds between CL and $\mathcal{BA}$, but not between CL and its "unofficial" semantics $\mathcal{HA}$.

Notational conventions

Given an equation $\alpha \approx \beta$ and a set of formulas in two variables $\rho = \{\alpha_j(p,q)\}_{j \in J}$, we use the abbreviation

$$\rho(\alpha, \beta) = \{\alpha_j(p/\alpha, q/\beta)\}_{j \in J}.$$

ρ will be also regarded as a function mapping equations to sets of formulas. If Γ, Δ are sets of formulas, $\Gamma \vdash_{\mathrm{L}} \Delta$ means $\Gamma \vdash_{\mathrm{L}} \alpha$ for all $\alpha \in \Delta$; if E, E' are sets of equations, $E \vdash_{Eq(\mathcal{K})} E'$ means $E \vdash_{Eq(\mathcal{K})} \epsilon$ for all $\epsilon \in E'$.

Algebraizable logics (1)

A logic $\mathrm{L} = (\mathbf{Fm}, \vdash_{\mathrm{L}})$ is said to be *algebraizable* with *equivalent algebraic semantics* $\mathcal{K}$ (where $\mathcal{K}$is a class of algebras of the same language as **Fm**) iff there exist a map τ from formulas to sets of equations, and a map ρ from equations to sets of formulas such that the following conditions hold for any $\alpha, \beta \in Fm$:

AL1: $\Gamma \vdash_{\mathrm{L}} \alpha$ iff $\tau(\Gamma) \vdash_{Eq(\mathcal{K})} \tau(\alpha)$;

AL2: $E \vdash_{Eq(\mathcal{K})} \alpha \approx \beta$ iff $\rho(E) \vdash_{\mathrm{L}} \rho(\alpha, \beta)$;

AL3: $\alpha \dashv\vdash_{\mathrm{L}} \rho(\tau(\alpha))$;

AL4: $\alpha \approx \beta \dashv\vdash_{Eq(\mathcal{K})} \tau(\rho(\alpha, \beta))$.

Algebraizable logics (2)

The sets $\tau(p)$ and $\rho(p, q)$ are respectively called a *system of defining equations* and a *system of equivalence formulas* for L and $\mathcal{K}$.
A logic L is algebraizable (tout court) iff, for some $\mathcal{K}$, it is algebraizable with equivalent algebraic semantics $\mathcal{K}$.
This definition can be drastically simplified: L is algebraizable with equivalent algebraic semantics $\mathcal{K}$ iff it satisfies either AL1 and AL4, or else AL2 and AL3.

Algebraizability of classical logic

Theorem

CL *is algebraizable with equivalent algebraic semantics* $\mathcal{BA}$*.*

Proof.

Let $\tau(p) = \{p \approx 1\}$ and $\rho(p, q) = \{p \to q, q \to p\}$. We need only check that τ and ρ satisfy conditions AL1 and AL4. However, we already proved AL1. As for AL4,

$$\begin{aligned} \alpha \approx \beta \dashv\vdash_{Eq(\mathcal{BA})} \tau(\rho(\alpha, \beta)) \quad &\text{iff} \quad \alpha \approx \beta \dashv\vdash_{Eq(\mathcal{BA})} \tau(\alpha \to \beta, \beta \to \alpha) \\ &\text{iff} \quad \alpha \approx \beta \dashv\vdash_{Eq(\mathcal{BA})} \{\alpha \to \beta \approx 1, \\ &\qquad\qquad \beta \to \alpha \approx 1\}. \end{aligned}$$

However, given any $\mathbf{A} \in \mathcal{BA}$ and any $\overrightarrow{a} \in A^n$, $\alpha^{\mathbf{A}}(\overrightarrow{a}) = \beta^{\mathbf{A}}(\overrightarrow{a})$ just in case $\alpha \to \beta^{\mathbf{A}}(\overrightarrow{a}) = 1^{\mathbf{A}}$ and $\beta \to \alpha^{\mathbf{A}}(\overrightarrow{a}) = 1^{\mathbf{A}}$, which proves our conclusion. □

Properties of algebraizability

- Every equivalent algebraic semantics for L is, in particular, an algebraic semantics for L in virtue of AL1. The converse need not hold.
- If L is algebraizable with equivalent algebraic semantics $\mathcal{K}$, then $\mathcal{K}$ might not be the unique equivalent algebraic semantics for L. However, in case L is finitary, any two equivalent algebraic semantics for L generate the same *quasivariety*. This quasivariety is in turn an equivalent algebraic semantics for the same logic (*the* equivalent quasivariety semantics for L).
- It is possible to have different algebraizable logics with the same equivalent algebraic semantics; however, if L and L' are algebraizable with equivalent quasivariety semantics $\mathcal{K}$ and with the *same set* of defining equations $\tau(p)$, then L and L' must coincide.

Algebraizability: a syntactic characterization

Theorem

A logic $\mathrm{L} = (\mathbf{Fm}, \vdash_{\mathrm{L}})$ *is algebraizable iff there exist a set* $\rho(p,q)$ *of formulas in two variables and a set of equations* $\tau(p)$ *in a single variable such that, for any* $\alpha, \beta, \gamma \in Fm$*, the following conditions hold:*

1. $\vdash_{\mathrm{L}} \rho(\alpha, \alpha)$;
2. $\rho(\alpha, \beta) \vdash_{\mathrm{L}} \rho(\beta, \alpha)$;
3. $\rho(\alpha, \beta), \rho(\beta, \gamma) \vdash_{\mathrm{L}} \rho(\alpha, \gamma)$;
4. *For every n-ary connective* c^n *and for every* $\overrightarrow{\alpha}, \overrightarrow{\beta} \in Fm^n$,

$$\rho(\alpha_1, \beta_1), \ldots, \rho(\alpha_n, \beta_n) \vdash_{\mathrm{L}} \rho\left(c^n\left(\overrightarrow{\alpha}\right), c^n\left(\overrightarrow{\beta}\right)\right)$$

5. $\alpha \dashv\vdash_{\mathrm{L}} \rho(\tau(\alpha))$.

In this case $\rho(p,q)$ *and* $\tau(p)$ *are, respectively, a set of defining equations and a set of equivalence formulas for* L.